Bifurcation analysis of an equation for gas discharge (Mathematical Analysis in Fluid and Gas Dynamics)

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1 Introduction

The main purpose of this paper is to analyze mathematically the fundamental mechanism of gas ionization processes. Nowadays plasmas are widely applied in various fields such as environmental improvement and engineering. The environmental applications are the production of ozone from air with water cleaning and the elimination of biological contamination. Engineers use plasma for material processing and surface deposition, electromagnetic absorbers and reflectors, and so on. Therefore, interest in the study of plasma generation has been increasing.

Townsend discovered the fundamental mechanism of gas ionization around 1900. He experimented and considered what happens in a chamber formed from two planar parallel plates and filled with a gas when a direct current high-voltage is applied between these two plates. Here the lower voltage plate is the cathode, and another one is the anode. As a consequence, it was observed that there are two mechanisms for a gas ionization process. If electrons are emitted by irradiation of X-rays to the cathode, these initial electrons are accelerated from the cathode to the anode by high-voltage and simultaneously make ions and additional electrons owing to the collision of electrons with gas particles. This mechanism is called as α-mechanism. Another mechanism, called as γ-mechanism, is the secondary emission of electrons caused by impact of positive ions with the cathode. If applied voltage is sufficiently high, these two mechanisms lead to the electric multiplication which permit large current flow throughout the gas which is an insulator. This phenomenon is called as avalanche breakdown or gas discharge. Townsend also derived a

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threshold of voltage at which gas discharge happens continuously. The threshold is called as sparking voltage. In this process, he used several simplification such as discretization of time, ignorance of advection, and so on (for more details, see [14]). Hence, it is an interesting problem to analyze the sparking voltage by using a partial differential equation with no simplification.

Morrow derived the mathematical model in [11]. After that, several models were proposed and used in [1, 5, 6, 7, 8, 9]. These models vary with the constitutive equations of velocities. On the other hand, Degond and Lucquin-Desreux in 2007 gave the formal derivation of the model, derived by Morrow, from the Euler–Maxwell equations (see [4]). At this point, it seems reasonable to analyze this model. In this paper, we call it as the Degond–Lucquin-Desreux–Morrow model. It consists of two continuity equations for the densities of positive ions and of electrons, adopting constitutive velocity relations, coupled with the Poisson equation for the electrostatic potential:

$$\partial_t \rho_i + \partial_x(\rho_i u_i) = a \exp\left(-b|\partial_x \Phi|^{-1}\right) \rho_e \left|v_e\right|, \tag{1a}$$

$$\partial_t \rho_e + \partial_x(\rho_e v_e) - k_e \partial_{xx} \rho_e = a \exp\left(-b|\partial_x \Phi|^{-1}\right) \rho_e \left|v_e\right|, \tag{1b}$$

$$\lambda \partial_{xx} \Phi = \rho_i - \rho_e. \tag{1c}$$

$$v_e := -k_e \partial_x \Phi, \quad u_i := k_i \partial_x \Phi. \tag{1d}$$

The unknown functions ρ_i, ρ_e, and $-\Phi$ denote the positive ion density, the electron density, and the electrostatic potential, respectively. The ion and electron velocities u_i and u_e are assumed to obey (1d). Moreover, k_i, k_e, a, b, and λ are positive constants. The right hand sides of (1a) and (1b) come from α-mechanism. In particular, $\alpha = a \exp\left(-b|\partial_x \Phi|^{-1}\right)$ is the first Townsend ionization coefficient expressing the number of ion–electron pairs generated per unit volume by the electron impact ionization. We notice that this model is a hyperbolic-parabolic-elliptic coupled system by substituting constitutive velocity relations (1d) into continuity equations (1a) and (1b).

We consider the initial–boundary value problem of this model over the bounded interval $I := (0, L)$ by prescribing the initial and boundary data

$$(\rho_i, \rho_e)(0, x) = (\rho_{i0}, \rho_{e0})(x), \quad \rho_{i0}(x) \geq 0, \quad \rho_{e0}(x) \geq 0, \quad x \in I, \tag{1e}$$

$$\rho_i(t, 0) = \rho_e(t, 0) = \Phi(t, 0) = 0, \tag{1f}$$

$$\rho_e(t, L) = 0, \;\; \Phi(t, L) = V_c > 0. \tag{1g}$$

The boundaries $x = 0$ and $x = L$ correspond to the anode and cathode, respectively, since $-\Phi$ is the electrostatic potential. Boundary condition (1f) means that, in an instant, electrons are absorbed to the anode and ions are excluded near the anode. We emphasize that γ-mechanism is not taken into account on the cathode $x = L$, and thus the zero Dirichlet boundary condition is adopted. From physical point of view, it is reasonable to assume the non-negativity of initial densities ρ_{i0} and ρ_{e0}. For the compatibility, we let the initial data R_{i0} and R_{e0} satisfy

$$R_{i0}(0) = R_{e0}(0) = R_{e0}(L) = 0.$$

For the Degond–Lucquin-Desreux–Morrow model and the related models, there are a lot of numerical researches (for example, see [10, 12, 13]). On the other hand, only two mathematical results for this model has been announced by the authors. The first result [15] established a mathematical framework for analyzing this model rigorously. More precisely, they showed the time-local solvability of the initial boundary value problem over a domain $\Omega := \mathbb{R}^3_+ \backslash K$, where $\mathbb{R}^3_+$ is a half space, K is a simply connected open set, and the intersection of $\partial\mathbb{R}^3_+$ and K is the empty set. The second result [16] investigated the sparking voltage, which is required when gas discharge happens continuously, by using (1). Specifically, the authors analyzed the bifurcation of the stationary solutions to (1) and then concluded that gas discharge may happen even for the no γ-mechanism case. This conclusion essentially differs from Townsend theory, since his theory explains that no gas discharge happens if there is no γ-mechanism. In this short paper, we review this bifurcation analysis. Before closing this section, we give a notation.

Notation For $1 \le p \le \infty$, $L^p(\Omega)$ is the Lebesgue space equipped with the norm $|\cdot|_p$. For a non-negative integer k, $H^k(I)$ is the k-th order Sobolev space in L^2 sense, equipped with the norm $\|\cdot\|_k$. Moreover, $H^1_0(I)$ and $H^1_{0l}(I)$ are closures of $C^\infty_0(I)$ and $C^\infty_0((0, L])$ with respect to H^1-norm, respectively. We denote by $C^m([0,T];X)$ the space of the m-times continuously differentiable functions on the interval $[0,T]$ with values in a Banach space X, and by $H^m(0,T;X)$ the space of H^m–functions on $(0,T)$ with values in a Banach space X.

2 Main results

For mathematical convenience, let us rewrite initial–boundary value problem (1) by using the new unknown functions

$$R_i := \rho_i e^{-\frac{L}{V_c}x}, \quad R_e := \rho_e e^{\frac{V_c}{2L}x}$$

and the new given functions

$$h(x) := a \exp\left(\frac{-b}{|x|}\right)|x|, \quad g(V_c) := h\left(\frac{V_c}{L}\right) - \frac{V_c^2}{4L^2}.$$

Furthermore, we also decompose the electrostatic potential as

$$\Phi = V + \frac{V_c}{L}x,$$

where $V_c x/L$ is a solution to the equation $\partial_{xx} u = 0$ with the boundary conditions $u(0) = 0$ and $u(L) = V_c$. As a result, we have the following rewritten problem

$$\partial_t R_i + k_i \partial_x \left\{ \left(\partial_x V + \frac{V_c}{L} \right) R_i \right\} + k_i R_i = k_e h \left(\frac{V_c}{L} \right) e^{-\frac{L}{V_c}x - \frac{V_c}{2L}x} R_e + k_i f_i, \tag{2a}$$

$$\partial_t R_e - k_e \partial_{xx} R_e - k_e g(V_c) R_e = k_e f_e, \tag{2b}$$

$$V[R_i, R_e] := \frac{1}{\lambda} \int_0^L G(x,y) \left(e^{\frac{L}{V_c}y} R_i(t,y) - e^{-\frac{V_c}{2L}y} R_e(t,y) \right) dy, \tag{2c}$$

$$(R_i, R_e)(0,x) = (R_{i0}, R_{e0})(x), \quad R_{i0}(x) \geq 0, \; R_{e0}(x) \geq 0, \tag{2d}$$

$$R_i(t,0) = R_e(t,0) = R_e(t,L) = 0, \tag{2e}$$

where $G(x,y)$ is the Green function of the Laplace operator with the Dirichlet zero condition, and the nonlinear terms f_i and f_e are defined as

$$f_i := - R_i \partial_x V - \frac{k_e}{k_i} \left\{ h \left(\frac{V_c}{L} \right) - h \left(\partial_x V + \frac{V_c}{L} \right) \right\} e^{-\frac{L}{V_c}x - \frac{V_c}{2L}x} R_e,$$

$$f_e := \partial_x V \partial_x R_e - \frac{V_c}{2L} R_e \partial_x V + R_e \partial_{xx} V - \left\{ h \left(\frac{V_c}{L} \right) - h \left(\partial_x V + \frac{V_c}{L} \right) \right\} R_e.$$

It is easy to check that the corresponding stationary problem has a trivial stationary solution

$$(R_i, R_e) = (0, 0).$$

The advantage of using the new known functions R_i and R_e lies in the following two facts. The first one is that the rewritten hyperbolic equation has the dissipative term $k_i R_i$, although the original hyperbolic equation does not have any dissipative structure. Secondly, the linear part of the rewritten parabolic equation is self-adjoint. These two facts play important roles in the proofs of both the nonlinear stability and instability of the trivial stationary solution.

We are now in a position to state the stability and instability theorems for the trivial solution.

Theorem 1. *Let $g(V_c) < \pi^2/L^2$. There exists $\varepsilon > 0$ such that if the initial data $(R_{i0}, R_{e0}) \in H^1_{0l} \times H^1_0$ satisfies $\|R_{i0}\|_1 + \|R_{e0}\|_1 < \varepsilon$, then problem (2) has a unique time global solution (R_i, R_e) as*

$$R_i \geq 0, \quad R_i \in C([0,\infty); H^1_{0l}) \cap C^1([0,\infty); L^2), \tag{3a}$$

$$R_e \geq 0, \quad R_e \in C([0,\infty); H^1_0) \cap L^2(0,\infty; H^2) \cap H^1(0,\infty; L^2). \tag{3b}$$

Moreover, it converges to zero exponentially fast in $H^1 \times H^1$ as t goes to infinity.

Theorem 2. *Let $g(V_c) > \pi^2/L^2$ and $(\psi_i, \psi_e) \in H^1_{0l} \times H^1_0$ satisfy*

$$\psi_i, \psi_e \geq 0, \quad \|\psi_i\|_1^2 + \|\psi_e\|_1^2 = 1, \quad \int_0^L \psi_e \sin \frac{\pi}{L} x \, dx > 0. \tag{4}$$

There exists $\varepsilon > 0$ such that for any sufficiently small $\delta > 0$, problem (2) with the initial data $(R_{i0}, R_{e0}) = (\delta\psi_i, \delta\psi_e)$ has a unique solution (R_i, R_e) satisfying $\|R_i(T)\|_1 + \|R_e(T)\|_1 \geq \varepsilon$ for some $T > 0$.

In this instability theorem, the last inequality in (4) is equivalent to that the initial data R_{e0} is a non-zero function. One may ask what happens for the case that R_{e0} is the zero function. Proposition 3 gives the answer that there exists a unique time global solution, and it attains the trivial stationary solution at finite time.

Proposition 3. *Let $V_c > 0$. There exists $\varepsilon > 0$ such that if the initial data $(R_{i0}, R_{e0}) \in H^1_{0l} \times H^1_0$ satisfies $R_{e0} = 0$ and $\|R_{i0}\|_1 < \varepsilon$, then problem (2) has a unique time global solution (R_i, R_e) as (3). Furthermore, there exists $T_0 > 0$ such that*

$$(R_i, R_e)(t, x) = (0, 0) \quad \text{for} \quad (t, x) \in [T_0, \infty) \times I. \tag{5}$$

Remark 4. *This proposition asserts that a set $\{(R_{i0}, R_{e0}) \in H^1_{0l} \times H^1_0\, ;\, R_{e0} = 0\}$ is a local stable manifold of system (2a)–(2c) for any $V_c > 0$.*

We can expect from Crandall and Rabinowitz's Theorem (see [2, 3]), and Theorems 1 and 2 by regarding the voltage V_c as the bifurcation parameter that there is a non-trivial solution curve near the point $(R_i, R_e, V_c) = (0, 0, V_c^*)$, where V_c^* is defined as

$$g(V_c^*) = \frac{\pi^2}{L^2}, \quad g'(V_c^*) > 0. \tag{6}$$

It is straightforward to check that the graph of the function g is drown as Figures 1 and 2, where there exists two cases subject to the physical parameters a, b, and L. For the first case as Figure 1, it has one local minimum and one global maximum. For the second case as Figure 2, it is strictly decreasing. Note that both cases are truly possible. We study only the first case hereafter and thus see that V_c^* in (6) is well-defined.

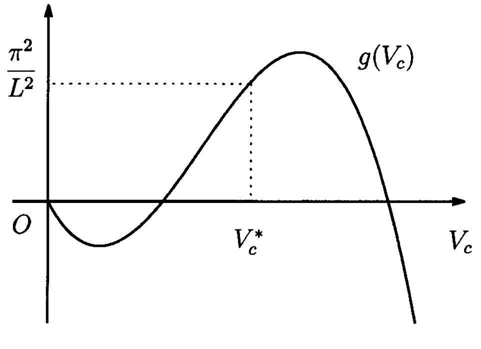

Figure 1: case 1

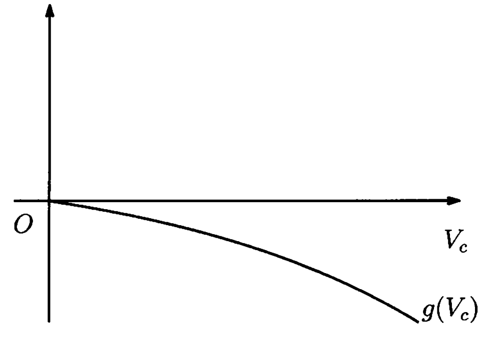

Figure 2: case 2

The bifurcation results are summarized in Theorem 5 and Corollary 6.

Theorem 5. *For V_c^* defined in (6), there exist $\eta > 0$, $V_c \in C^2([-\eta,\eta];\mathbb{R})$, and $z \in C^2([-\eta,\eta]; H^1 \times H^2)$ such that $V_c(0) = V_c^*$, $z(0) = 0$, and stationary problem to (2) with $V_c = V_c(s)$ has a non-trivial solution $(R_i, R_e)(s) = s(\varphi_i, \varphi_e) + sz(s)$ for $s \in [-\eta, \eta]$, where*

$$\varphi_i(x) := \frac{k_e}{k_i} \exp\left(\frac{-bL}{V_c^*}\right) e^{-\frac{L}{V_c^*}x} \int_0^x e^{-\frac{V_c^*}{2L}y} \varphi_e(y)\, dy, \quad \varphi_e(x) := \sin \frac{\pi}{L} x.$$

Moreover, $\dot{V}_c(0) \lesseqgtr 0$ holds if and only if

$$-Lg'(V_c^*) \int_0^L \varphi_e^2 \partial_x V[\varphi_i, \varphi_e]\, dx - \frac{1}{2} \int_0^L \varphi_e^2 \partial_{xx} V[\varphi_i, \varphi_e]\, dx \lesseqgtr 0.$$

Corollary 6. *Let $\dot{V}_c(0) \neq 0$. For $s \neq 0$, it holds for any $x \in (0, L)$ that*

$$s\dot{V}_c(0) R_i(s) > 0, \quad s\dot{V}_c(0) R_e(s) > 0.$$

Furthermore, the positive non-trivial solution is linearly stable if $\dot{V}_c(0) > 0$, and the positive non-trivial solution is linearly unstable if $\dot{V}_c(0) < 0$.

From Theorem 5 and Corollary 6, we can draw the bifurcation diagram of stationary solutions as Figures 3 and 4. The both diagrams are truly possible for some physical parameters k_i, k_e, a, b, and L. For example, one can have $\dot{V}_c(0) > 0$ by letting k_e/k_i sufficiently small; one can have $\dot{V}_c(0) < 0$ by letting k_e/k_i sufficiently large and assuming an additional condition for a, b, and L.

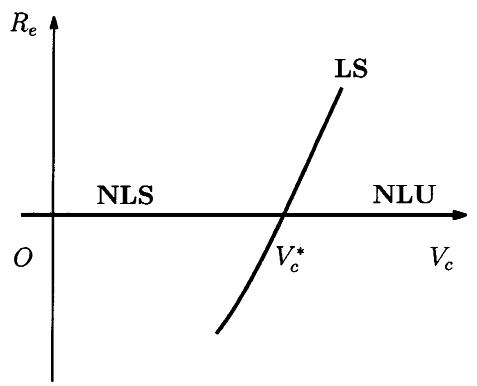

Figure 3: Case $\dot{V}_c(0) > 0$

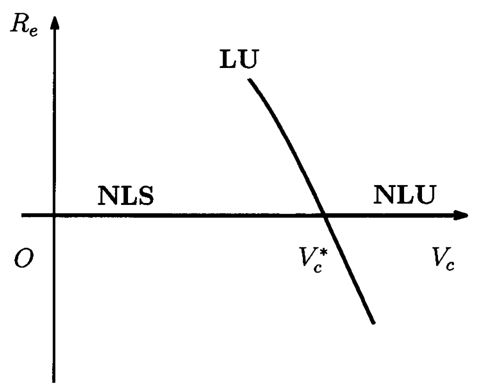

Figure 4: Case $\dot{V}_c(0) < 0$

Let us mention physical observation from the above bifurcation analysis. Townsend defined the sparking voltage as a threshold of voltage at which gas discharge happens continuously. In the following his manner, it is reasonable to define the sparking voltage for the Degond–Lucquin-Desreux–Morrow model by V_c^* in (6). In fact, for the case $V_c > V_c^*$, the solution to this model may approach to the positive non-trivial stationary solution as t tends to infinity if $\dot{V}_c(0) > 0$; the solution may either blow up or grow up as time goes by if $\dot{V}_c(0) < 0$. Hence, the solution never goes to the trivial stationary solution $(R_i, R_e) = (0, 0)$. On the other hand, for the case $V_c < V_c^*$, the solution converges to the trivial solution as t tends to infinity. These facts mean that V_c^* is a threshold of voltage at which gas discharge happens continuously from physical point of view. Therefore we can conclude that gas discharge can happen even if γ-mechanism is not taken into account, whereas it cannot happen without γ-mechanism in Townsend theory.

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